Blog Post 4

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Introduction:

Hey guys, so this is a shorter, more informal post about a basic mathematical topic. In the past day, one of my friends asked me for help on an interesting calculus problem shown below:

If f and g are inverse functions, then:

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

At first, when I looked at it, I thought the problem itself was incredibly interesting since it did not seem at all intuitive. Anyway, I worked on it for about ten minutes, trying everything from u-sub, to the mean-value-theorem to the fundamental theorem of calculus looking for some sort of solution. After about 15 minutes, at last I realized, it was the bane of all calculus, integration by parts! The reason I say integration by parts is the "bane of calculus" is because its hard to visualize integration by parts, and the theorem itself seems to come out of no where. The same is true for the solution to this problem which I will now present.

Proof using Integration By Parts

To prove this theorem, consider $u = f(x)$ and $dv = 1$ (the function 1). Therefore $v = x$. Then, performing integration by parts on u and dv we have the following:

$$\int_a^b f(x)dx = uv - \int vdu = xf(x)|_a^b - \int_a^b xf'(x)dx$$

Now, we use a u-sub, except my u will be y. So we have $y = f(x)$, then $dy = f'(x)dx$. Here is the punchline: Since $g(f(x)) = x$, we see that $g(y) = x$. So we can finish the substitution of the last integral as $\int_{f(a)}^{f(b)} g(y)dy$, so we have in total:

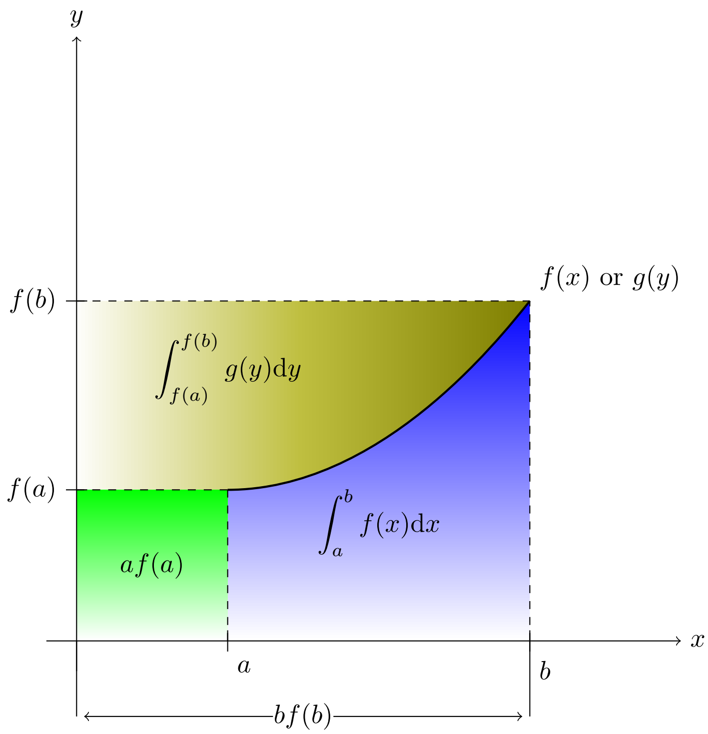
$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

A Nagging Feeling

So yeah, I finished the proof and gave my friend hints until she was able to figure it out for herself, and that should have been the end of it right?. Well, I had a nagging feeling that something else was going on. It did not seem so likely that integration by parts out of all things could solve something that looked so elegant and bizarre. I figured there had to be a visual proof that relates the integral of a function to the integral of its inverse right!?

The next day, I was in the shower, and I was lucky to have a window next to me. So I steamed up the shower and started drawing on the window creating picture to help me understand what was going on. It was here I had an epiphany which I want to share with you guys.

A Visual Proof (Proof Without Words)



From here we see that

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

Conclusion

The above proof is not only simple, but beautiful and it avoids the non-visual pitfalls of using integration by parts. Consequently, this problem illustrates an important phenomena in mathematics: visualization. Visualization is important because it helps connect ideas within mathematics. As humans, we learn through visual effects, and not through formulas. Writing formulas as visual ideas can not only help us strengthen our understanding of the formulas, but also help us explore deeper ideas. The theorem itself relates the notion of inverses to their integrals, but the visualization shows us how the graphical notion of inverses allows for this theorem to be true. Additionally, this visualization strengthens the notion of integration with the notion of area, since by relating the two terms in this way, we can get more familiar with how integrals of inverse functions operate. Finally, the visualization avoids the pitfalls of formulaic mathematics such as the 'integration by parts formula' and demonstrates a deeper, and frankly cooler idea behind the formula.